$$= \prod_{i=m}^{n-1}(1-P_f)^{\xi_i} \cdot \prod_{i=m+1}^{n-1}(1-\sum_{j=m}^{i-1}\gamma_j)^{\xi_i}$$

$$= (1-P_f)^{(\lambda_m - \alpha_2^*)} \cdot \prod_{i=m+1}^{n-1}(1-\sum_{j=m}^{i-1}\gamma_j)^{\xi_i},$$

where $\alpha_2^* = \sum_{i=m+1}^{n+1}\alpha_i$,where $\alpha_{n+1} = \lambda_n$ as we mentioned before.

Similarly, to show necessity, assume that perfect aggregation holds with this prior. Then $\prod_{i=1}^{n-1}(1-\sum_{j=1}^{i} p_j)^{\xi_i}$ must be factorable in to a function of P_f and a function of γ (theorem 7.2), which cannot be true unless condition (7.10) holds. One can see that by contradiction; i.e., by assuming that at least one of the ξ_i shown in condition (7.10) is not equal to zero.

Collecting terms, we get

$$g(P) \sim g_1(P_f)\cdot g_2(\gamma)\cdot \prod_{i=1}^{n-1}(1-\sum_{j=1}^{i} p_j)^{\xi_i}$$

$$= P_f^{\sum_{i=1}^{m}\alpha_i - m}(1-P_f)^{\lambda_m-(n-m+1)}\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}(1-\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\cdot$$

$$\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}(1-\sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1}-1}\prod_{i=m+1}^{n-1}(1-\sum_{j=m}^{i-1}\gamma_j)^{\xi_i}$$

$$= [P_f^{\sum_{i=1}^{m}\alpha_i - m}(1-P_f)^{\lambda_m-(n-m+1)}].\left[\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}(1-\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\right].$$

$$\left[\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}(1-\sum_{j=m}^{i}\gamma_j)^{\xi_{i+1}}\right]$$

Hence, within this class of distributions, condition (7.10) is necessary and sufficient for perfect aggregation to hold.

Remark 7.2

i) Wong (1996) proved that if the prior distribution of the system state probabilities is a Connor-Mosimann distribution $CM_n(\alpha,\lambda)$, then condition (7.10) is necessary and sufficient for perfect aggregation to hold. However, comparing our approach to his, one can see the usefulness of theorem 7.2 as a means to discover easily whether a prior distribution satisfies the conditions for perfect aggregation, without the extensive algebra used in his approach. Our approach is also potantially applicable to a much wider class of distributions.

ii) The joint density $h(P_f, \gamma)$ is given by

$$h(P_f,\gamma) \sim P_f^{\,m-1}(1-P_f)^{n-m}[P_f^{\,\sum_{i=1}^{m}\alpha_i - m}(1-P_f)^{\lambda_m-(n-m+1)}]\left[\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}(1-\textstyle\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\right]\cdot$$

$$\left[\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}(1-\textstyle\sum_{j=m}^{i}\gamma_j)^{\xi_{i+1}}\right]$$

$$=\underbrace{[P_f^{\,\alpha_1^*-1}(1-P_f)^{\lambda_m-1}]}_{Be(\alpha_1^*,\lambda_m)}\underbrace{\left[\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}(1-\textstyle\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\right]}_{D_m(\alpha_1,\alpha_2,\cdots,\alpha_m)}\underbrace{\left[\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}(1-\textstyle\sum_{j=m}^{i}\gamma_j)^{\xi_{i+1}}\right]}_{CM_{n-m}(\cdot,\cdot)},$$

where $\alpha_1^* = \sum_{i=1}^{m}\alpha_i$.

iii) Note that the condition for perfect aggregation ($\xi_i = 0$ for i=1,..,m-1) that was given in (7.10) is related to Ψ, the set of failure states. So one may ask, is there a condition related to Ω/Ψ, the set of success states, such that perfect aggregation holds? The answer is yes. All we need is to change the order of the random variables representing the system probabilities, since in a generalized Dirichlet distribution this order is not arbitrary. For example, if we let $(p_{n+1}, p_n, \ldots, p_2, p_1)$ follow a generalized Dirichlet distribution with density function given by

$$g(p_{n+1},\ldots,p_2)=\prod_{i=1}^{n}B^{-1}(\alpha_{n+2-i},\lambda_{n+2-i})\,p_{n+2-i}^{\alpha_{n+2-i}-1}(1-\textstyle\sum_{j=1}^{i}p_{n+2-j})^{(\lambda_{n+2-i}-\alpha_{n+1-i}-\lambda_{n+1-i})},$$

then the condition for perfect aggregation is $\lambda_{n+2-i}=\alpha_{n+1-i}+\lambda_{n+1-i}$, $\quad i=1,2,\ldots,n-m$.

Before proceeding to our third example, we first define the Aitchison distribution, $A^n(\alpha,\boldsymbol{B})$. The definition of this class of distributions and the remark following it are due to Aitchison (1986, chapter 13).

Definition 7.1 A random vector $\mathbf{x}\in S^n$ is said to have the $A^n(\alpha,\boldsymbol{B})$ distribution if **x** has probability density function $g_A(\mathbf{x}\,|\,\alpha,\boldsymbol{B})$, defined by

$$\log[g_A(\mathbf{x}\,|\,\alpha,\boldsymbol{B})]=k(\alpha,\boldsymbol{B})+\sum_{i=1}^{n+1}(\alpha_i-1)\log(x_i)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}b_{ij}[\log(x_i)-\log(x_j)]^2,$$

where the parameters α, $\boldsymbol{B}$ must satisfy either

(a) $\alpha_1+\alpha_2+\ldots+\alpha_{n+1}\geq 0$ and positive definiteness of the matrix ***B***,

or

(b) $\alpha_i>0$ $(i=1,2,\ldots n+1)$ and non-negative definiteness of the matrix ***B***.

Remark 7.3

If $b_{ij}=0\,(i=1,2,\ldots,n;\ j=i+1,\ldots,n+1)$ and $\alpha_i>0$ $(i=1,2,\ldots n+1)$, then the Aitchison distribution reduces to the Dirichlet distribution. Also, the Aitchison distribution reduces to the additive logistic normal distribution when $\alpha_1+\alpha_2+\ldots+\alpha_{n+1}=0$.

Example 7.3 (*Aitchison distribution,* $A^n(\alpha,\boldsymbol{B})$)

Let the prior joint density g of P be $A^n(\alpha,\boldsymbol{B})$ as defined above. Then perfect aggregation holds for our model if the b_{ij} satisfy the following condition:

$$b_{ij} = 0, \quad i = 1,\ldots,m, \quad j = m+1,\ldots,n+1. \tag{7.11}$$

One can show this as in the previous examples (i.e., by using theorem 7.2), or see Aitchison (1986, p. 316).

Note that the conditions for perfect aggregation are more general than the conditions under which the Aitchison distribution reduces to the Dirichlet. Thus, this result yields a new class of distributions that satisfies perfect aggregation. However, the usefulness of this class in practice is likely to be limited, since Aitchison speculates that the normalizing term $k(\alpha, \boldsymbol{B})$ is not closed form except for the special cases of the Dirichlet and additive logistic normal distributions.

Example 7.4 (***CGL*** *distribution*)

If the prior joint density $g(P)$ is conditional generalized Liouville, as given in definition 4.4, then perfect aggregation holds for our model iff $\beta_i = 1$ for i=1,...n+1, $q_1 = \ldots = q_m$, and $q_{m+1} = \ldots = q_{n+1}$. This can be shown as in the previous examples (i.e., by using theorem 7.2), or see Smith (1994, theorem 9.2).

Note that before we gave specific definitions for P_f and γ in terms of $P = (p_1, p_2, \ldots, p_n)$ and theorem 7.2, we had no means to discover easily whether the above priors satisfied perfect aggregation. As stated before, Hazen's result by itself is usually not helpful in verifying whether perfect aggregation holds for a particular choice of prior distribution. However, Hazen's result provided the basis for our result.

7.5.2 New distributions that satisfy perfect aggregation

As stated in chapter 1 and section 3.4, one of our concerns in this work is to identify a family of distributions that contains the Dirichlet but also contains distributions capable of modeling nontrivial dependence structures and still satisfying perfect aggregation. We already developed some such distributions in chapters 5 and 6. In this section, we introduce some additional families of distributions that facilitate studying perfect aggregation. In particular, this section introduces two approaches for developing such families of distributions. Unfortunately, some of our new distributions are not analytically tractable. However, one can use Monte Carlo simulation or numerical integration to calculate moments and other performance measures.

One approach for developing new families of distributions on the n-dimensional unit simplex is to assign a simplex distribution of lower dimensionality to each element of a partition of order 1 of the original n-dimensional vector. For example, let **x** = $(x_1,\ldots,x_n) \in S^n$, and consider a partition of order one, $\{\boldsymbol{s}_1,\boldsymbol{s}_2,t\}$, where

$$\boldsymbol{s}_1 = \left(x_1/t,\ldots,x_{d-1}/t\right),\ \boldsymbol{s}_2 = \left(x_{d+1}/(1-t),\ldots,x_n/(1-t)\right), \text{ and } t = \textstyle\sum_{i=1}^{d} x_i\,. \tag{7.12}$$

Note that $\boldsymbol{s}_1 \in S^{d-1}$, $\boldsymbol{s}_2 \in S^{n-d}$, and $t \in S^1 = (0,1)$. Assume that $\mathbf{s}_1,\mathbf{s}_2$, and t are mutually independent (i.e., **x** possesses partition independence), and f_1, f_2, and f_3 are the densities of $\boldsymbol{s}_1,\boldsymbol{s}_2$, and t, respectively. Then the density of **x** is of the form

$$g(\mathbf{x}) = f_1(\boldsymbol{s}_1)\, f_2(\boldsymbol{s}_2)\, f_3(t)\, J((\boldsymbol{s}_1,\boldsymbol{s}_2,t)\to\mathbf{x})\,, \tag{7.13}$$

where $J((\boldsymbol{s}_1,\boldsymbol{s}_2,t)\rightarrow\mathbf{x}) = (\sum_{i=1}^{d} x_i)^{1-d}\,(1-\sum_{i=1}^{d} x_i)^{d-n}$ is the Jacobian of the transformation $(\boldsymbol{s}_1,\boldsymbol{s}_2,t)\rightarrow\mathbf{x}$ (i.e., the inverse of the transformation given by (7.12)). To illustrate this approach, consider the following examples:

1) If $\boldsymbol{s}_1 \sim D_{d-1}(\alpha_1,...,\alpha_{d-1};\alpha_d)$, $\mathbf{s}_2 \sim D_{n-d}(\alpha_{d+1},\ldots,\alpha_n;\alpha_{n+1})$, and $t \sim Be(\sum_{i=1}^{d}\alpha_1,\sum_{i=d+1}^{n+1}\alpha_i)$, then we have $\mathbf{x} \sim D_n(\alpha_1,\ldots\ldots,\alpha_n;\alpha_{n+1})$.

2) If $\boldsymbol{s}_1 \sim D_{d-1}(\alpha_1,...,\alpha_{d-1};\alpha_d)$, $\boldsymbol{s}_2 \sim CM_{n-d}(\alpha_{d+1},\ldots,\alpha_n;\lambda_{d+1},\ldots,\lambda_n)$, and $t \sim Be(\sum_{i=1}^{d}\alpha_i,\lambda_d)$, then we have $\mathbf{x} \sim CM_n(\alpha_1,\ldots,\alpha_n;\lambda_1,\ldots,\lambda_n)$, where $\lambda_i = \alpha_{i+1}+\lambda_{i+1}$ for $i = 1,\ldots,d-1$.

3) For n=3 (i.e., $x_1+x_2+x_3+x_4 = 1$), if $\boldsymbol{s}_1 = x_1/(x_1+x_2) \sim Be(\alpha_1,\lambda_1)$, $\boldsymbol{s}_2 = x_3/(x_3+x_4) \sim Be(\alpha_3,\lambda_3)$, and $t=x_1+x_2 \sim Be(\alpha_2,\lambda_2)$, then we have $\mathbf{x} \sim AD_3^{(B)}(\alpha_1,\alpha_2,\alpha_3;\lambda_1,\lambda_2,\lambda_3,2)$.

4) If $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ follow any simplex distributions, and t follows any distribution on (0,1), then we have a distribution for **x** on the unit simplex. For some choices of distributions for $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ (e.g., $\boldsymbol{s}_1 \sim$ adaptive Dirichlet with dependent ratios and $\boldsymbol{s}_2 \sim CGL$), the results will establish new simplex distributions.

The approach discussed here, as well as the specific definitions for P_f and γ in terms of $P = (p_1, p_2,\ldots, p_n)$ given in equations (7.7) and (7.8), will allow us to construct a large number of new prior distributions that satisfy the conditions of theorem 7.2 for perfect aggregation. For example, if states 1 through m in Ω are the states in which the system fails, one could let $(p_1/t, p_2/t,..., p_m/t) \sim$ adaptive Dirichlet with dependent ratios and $(p_{m+1}/t,..., p_n/t) \sim CGL$.

The idea above can be further generalized in one of two ways. First, one could start from the set $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{k+1}; \boldsymbol{t}\}$; i.e., a partition of order k instead of order 1. Alternatively, one could have a hierarchy of partitions. For example, if we have a partition of order one at level d (i.e., $\boldsymbol{s}_1 \in S^{d-1}$, $\boldsymbol{s}_2 \in S^{n-d}$, and $t = \sum_{i=1}^{d} x_i$), then we could partition $\boldsymbol{s}_1$ and/or $\boldsymbol{s}_2$ again, and so on. One should note that those multiple Dickey distributions obtained from successively nested partitions form a subclass of the resulting new family (although multiple Dickey distributions obtained from overlapping sets are not included in our new family). Again, by suitable choice of the partitions, many distributions established using these approaches will satisfy perfect aggregation.

We now present a second approach for developing new simplex distributions. The resulting new class of distributions contains as special cases the Dirichlet, Connor-Mosimann, CGL, and multiple Dickey distributions. Note also that this family contains new distributions satisfying the conditions for perfect aggregation.

Definition 7.2 $\mathbf{x} \in S^n$ is said to have a ***generalized multiple Dickey distribution***, $GMD_n(f; \alpha, \zeta, \beta, \boldsymbol{q}, \boldsymbol{C})$ if $\mathbf{x}$ has density

$$g(\mathbf{x}) = A\left[\prod_{i=1}^{n+1} x_i^{\alpha_i - 1}\right]\left[\prod_{j=1}^{L}\left(\sum_{i=1}^{n+1}(c_{ij}x_i)^{\beta_i}\right)^{\zeta_j}\right] f\left(\sum_{i=1}^{n+1}(x_i/q_i)^{\beta_i}\right) \tag{7.14}$$

where $x_{n+1} = 1 - x_1 - \cdots - x_n$, $\alpha_i > 0$, $q_i > 0$, $\beta_i > 0$, $\zeta_j \in \Re$, and $c_{ij} \geq 0$ for all i and j, L=1,2,…., and f is a positive continuous function such that

$$\frac{1}{A} = \int_{S^n}\left[\prod_{i=1}^{n+1} x_i^{\alpha_i - 1}\right]\left[\prod_{j=1}^{L}\left(\sum_{i=1}^{n+1}(c_{ij}x_i)^{\beta_i}\right)^{\zeta_j}\right] f\left(\sum_{i=1}^{n+1}(x_i/q_i)^{\beta_i}\right) < \infty .$$

Remark 7.4

(1) If $\beta_i = 1,\ q_i = q_j$, and $c_{ij} = 1$ for all *i,j* , then we get the Dirichlet distribution.

(2) If $\beta_i = 1,\ q_i = q_j$ for all *i,j*, *L*=*n-1*, and $c_{jk} = 0$, $c_{ik} = 1$ for all $j \le k$, $i > k$, $k = 1, \cdots, n-1$, and if there exist $\lambda_i > 0$ such that $\zeta_i = \lambda_i - \alpha_{i+1} - \lambda_{i+1}$, $i = 1, \cdots, n-1$, and $\lambda_n = \alpha_{n+1}$, then we get the Connor-Mosimann distribution.

(3) If L=1, $\zeta_1 = 1 - \sum_{i=1}^{n+1} \alpha_i / \beta_i$, and $c_{i1} = 1/q_i$ for all i, then we get the conditional generalized Liouville distribution.

(4) We cannot get either the Liouville or the generalized Liouville distributions on the unit simplex as special cases of this distribution. However, when $\beta_1 = \cdots = \beta_n = 1$, $q_1 = \cdots = q_n = 1$, and $\alpha_{n+1} = 1$, then the conditional generalized Liouville distribution approaches the standard Liouville distribution as $q_{n+1} \to \infty$.

(5) If $\beta_i = 1$ and $q_i = q_j$ for all *i,j*, then we get the multiple Dickey distribution.

(6) Perfect aggregation is achieved if $\beta_i = 1$ for i=1,...,n+1, $q_1 = \ldots = q_m$, $q_{m+1} = \ldots = q_{n+1}$, and each column of the matrix $\boldsymbol{C}$ has one of the following forms:

$$c_{*j} = \begin{cases} (\underbrace{c_j, c_j, \ldots, c_j}_{m}, 0, \ldots, 0)^t \\ \text{or} \\ (\underbrace{0, \ldots, 0}_{m}, b_j, \ldots, b_j)^t \end{cases} \quad \text{for } j = 1, \ldots, L, \tag{7.15}$$

where states 1 through m in Ω are the states in which the system fails. Note that these conditions are equivalent to the following new distribution that satisfies the conditions for perfect aggregation:

$$g(\mathbf{x}) = A\left[\prod_{i=1}^{n+1} x_i^{\alpha_i - 1}\right]\left[\prod_{j=1}^{m}\left(c_j \sum_{i=1}^{m} x_i\right)^{\zeta_j}\right]\left[\prod_{j=m+1}^{L}\left(c_j \sum_{i=m+1}^{n+1} x_i\right)^{\zeta_j}\right] f\left(1/q_1 \sum_{i=1}^{m} x_i + 1/q_2 \sum_{i=m+1}^{n+1} x_i\right).$$

Figure 7.1 shows the relations between the Dirichlet, Connor-Mosimann, conditional generalized Liouville, multiple Dickey, and generalized multiple Dickey distributions. Distributions in the dashed region satisfy perfect aggregation. Thus, perfect aggregation is satisfied by some but not all multiple Dickey, Connor-Mosimann, and conditional generalized Liouville distributions.

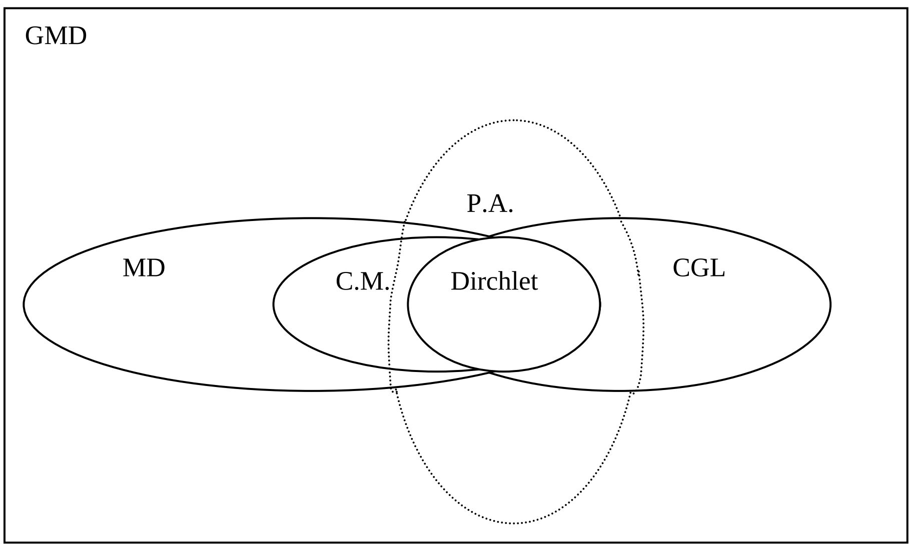

Figure 7.1 Relationship among families of simplex distribution.

$Dir \subset C.M. \subset MD \subset GMD \supset CGL \supset Dir$ and $MD \cap CGL = Dir$.

7.6 Perfect aggregation for parallel Poisson systems

The results of the previous sections can be extended to parallel Poisson systems (i.e., systems constructed by placing a component that fails in a Poisson manner with failure rate λ_0 in parallel with a Bernoulli system as defined in section 7.1.

For the parallel Poisson systems we will adopt Azaiez's assumptions (Azaiez, 1993), which are given below:

1) The system is observed for a preassigned time.

2) The Bernoulli system is taken to be stand-by or back-up for the Poisson component and as such is not tested unless the Poisson component fails.

3) As in section 7.2, it is assumed that we test all components of the Bernoulli system simultaneously.

4) Failure of one component or subsystem does not affect the failure probability of any other component, or subsystem.

Analogous to the reparametrization of the Bernoulli system discussed in section 7.4, the parameters of this model are defined as follows:

$P_\lambda = (\lambda_0, P)$, where P is defined as in section 7.4,

$\lambda_f = \lambda_0 \sum_{i=1}^{m} p_i$ (the system failure rate), and

$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, where $\gamma_1 = \lambda_0 - \lambda_f$, $\gamma_{i+1} = \dfrac{\lambda_0 p_i}{\lambda_f}$ for i= 1,...,m-1, and $\gamma_i = \dfrac{\lambda_0 p_i}{\lambda_0 - \lambda_f}$ for i = m+1 ,...,n.

Making the transformation $P_\lambda \xrightarrow{T} (\lambda_f, \gamma)$, which has Jacobian $\dfrac{\lambda_f^{\,m-1}(\lambda_0 - \lambda_f)^{n-m}}{\lambda_0^{\,n}}$, we can state the following theorems, which are analogous to theorems 7.1 and 7.2. The proofs are similar to the ones given for those theorems.

Theorem 7.3 Perfect aggregation holds for this model iff λ_f and γ are independent.

Theorem 7.4 Perfect aggregation holds for this model if and only if $\lambda_0^{-n} g_0(P_\lambda)$ can be factored into a function of λ_f and a function of γ, where g_0 is the prior joint density of P_λ.

For parallel Poisson systems defined above, Azaiez (1993) proved the following result:

If $\lambda_0 \sim Gamma(a,b)$ *and* $P \sim D_n(\alpha_1,\ldots,\alpha_n;\alpha_{n+1})$ *with* $a = \sum_{i=1}^{n+1}\alpha_i$ *, then perfect aggregation holds.*

This is a generalization of a result due to Bier (1993) for a simple system consisting of a single Poisson component and a single Bernoulli component. Example 7.5 below generalizes Azaiez's result by allowing *P* to have a Connor-Mosimann distribution instead of a Dirichlet distribution. Also, we speculate that there exist conditions on the distribution of λ_0 so that perfect aggregation holds for the entire Poisson system whenever perfect aggregation holds for the Bernoulli system.

Example 7.5

Let $\lambda_0 \sim Gamma(a,b)$ and $P \sim CM_n(\alpha,\eta)$ with prior density given by

$$g(p_1,p_2,\ldots,p_n) = \prod_{i=1}^{n} B^{-1}(\alpha_i,\eta_i)p_i^{\alpha_i-1}(1-\textstyle\sum_{j=1}^{i} p_j)^{\xi_i},$$

where $\xi_i = \eta_i - \alpha_{i+1} - \eta_{i+1}$ for i =1,2,...,n-1 and $\xi_n = \eta_n - 1$. Then perfect aggregation holds iff $\eta_i = \alpha_{i+1} + \eta_{i+1}$ for $i = 1,2,\ldots,m-1$ and $a = \eta_m + \sum_{i=1}^{m}\alpha_i$.

One can show this by proving that the given conditions hold iff $\lambda_0^{-n} g_0(P_\lambda)$ can be factored into a function of λ_f and a function of γ, where g_0 is the prior joint density of P_λ.

Note that if $P \sim D_n(\alpha_1,\ldots,\alpha_n;\alpha_{n+1})$ with α_i (i =1,2,...,n) as given above and $\alpha_{n+1} = \eta_n$, then $\eta_i = \alpha_{i+1} + \eta_{i+1}$ for $i = 1,\ldots,n-1$, which implies that $\eta_m = \sum_{i=m+1}^{n+1}\alpha_i$ and therefore $a = \sum_{i=1}^{n+1}\alpha_i$. So if $P \sim D_n(\alpha_1,\ldots,\alpha_n;\alpha_{n+1})$ and $\lambda_0 \sim Gamma(a,b)$, then perfect aggregation holds iff $a = \sum_{i=1}^{n+1}\alpha_i$. This coincides with theorem 8.3 given by Azaiez (1993), except that he gave $a = \sum_{i=1}^{n+1}\alpha_i$ as a sufficient condition only, while we give this as a necessary and sufficient condition for perfect aggregation to hold.

The examples above show the usefulness of our results. For example, we extended the results of Azaiez from the Dirichlet case to the case of Connor-Mosimann distributions, and were able to show the necessity of the sufficient condition Azaiez found in the Dirichlet case. However, since perfect aggregation is only one of our two application areas, we do not propose to extend these results for Poisson systems in the dissertation, and instead will restrict ourselves to studying perfect aggregation in Bernoulli systems.