As mentioned by Azaiez (1993) in the conclusion of his chapter 8, this result is usually not helpful in confirming whether perfect aggregation holds for a particular prior distribution, since Hazen's result does not tell us how to choose δ and γ. Also, this result is not helpful in constructing a prior distribution that satisfy perfect aggregation for the same reason. In section 7.4 below, for our Bernoulli system we will define particular choices of δ and γ in terms of $P = (p_1, p_2, \ldots, p_n)$ and show that if we have a prior density $g(P)$, then perfect aggregation holds if and only if g can be factored into a function of δ and a function of γ. Our suggested choices of δ and γ, when coupled with the approach discussed in chapter 6, will allow us to construct a large number of new prior distributions that satisfy perfect aggregation. Using the result of section 7.4, many examples of prior distributions for $P = (p_1, p_2, \ldots, p_n)$ that satisfy perfect aggregation will be given in section 7.5. A condition for perfect aggregation to hold in some Poisson systems (as opposed to Bernoulli systems) will then be given in section 7.6.

7.4 Conditions for perfect aggregation in Bernoulli systems

As stated earlier, Hazen's result is usually not helpful in verifying whether perfect aggregation holds for a particular prior distribution. However, in this section we will define a specific choice of δ and γ in terms of $P = (p_1, p_2, \ldots, p_n)$ and show that perfect aggregation holds for a Bernoulli system if and only if the specified δ and γ are independent. In addition, we will show that perfect aggregation holds if and only if the prior density function of P can be factored into a function of δ and a function of γ. Note also that our choice of δ and γ is unique.

Consider the Bernoulli system discussed in sections 7.1 and 7.2. Note that no distribution is assigned to $P = (p_1, p_2, \ldots, p_n)$ at this point. Now we want to define δ and γ so that conditions (7.5) and (7.6) hold.

Let $\delta = P_f = \sum_{i=1}^{m} p_i$ (the system failure probability), and (7.7)

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{n-1}), \tag{7.8}$$

where $\gamma_i = p_i / P_f$ for i = 1,2,...,m-1, and $\gamma_i = p_{i+1} / (1 - P_f)$ for i = m, m+1,...,n-1.

From the above, note that:

1) P is completely determined by (P_f, γ) and vice versa. Note for example that $p_m = P_f (1 - \sum_{i=1}^{m-1} \gamma_i)$.

2) The Jacobian of the transformation $P \rightarrow (P_f, \gamma)$ is $P_f^{\,m-1}(1 - P_f)^{n-m}$ for $1 \le m \le n$, which is a function of P_f.

3) $K_0 = \sum_{i=1}^{n+1} k_i$.

4) Saying that P_f and γ are independent is equivalent to saying that $P = (p_1, p_2, \ldots, p_n)$ possesses subcompositional invariance.

5) Since partition independence implies subcompositional invariance, then if $P = (p_1, p_2, \ldots, p_n)$ possesses partition independence, P_f and γ are independent.

Theorem 7.1 *Let* P_f *and* γ *be as defined above. Then perfect aggregation holds for our Bernoulli system if and only if* P_f and γ *are independent* (*i.e., if and only if* $P = (p_1, p_2, \ldots, p_n)$ *possesses subcompositional invariance with respect to the partition defined by* P_f *and* γ).

To prove theorem 7.1, we need to prove all the requirements in Hazen's result. This will be done in propositions 7.1 and 7.2.

Proposition 7.1 *If DD, AD, P,* P_f *, and* γ *are defined as above, then the conditions* $AD \mid (P_f\ ,\ \gamma) \overset{d}{=} AD \mid P_f$ and $DD \mid (AD,\ P_f\ ,\ \gamma) \overset{d}{=} DD \mid (AD,\ \gamma)$ *given in equations 7.5 and 7.6 hold.*

Proof

Clearly $AD \mid (P_f\ ,\gamma) \overset{d}{=} (AD \mid P_f) \sim Bin(K_0, P_f)$; i.e.,

$$P(K^* = k^* | P_f) = \binom{K_0}{k^*} P_f^{\ k^*} (1 - P_f)^{K_0 - k^*}.$$

So condition (7.5) holds. For condition (7.6) to hold, we need to show that

$DD \mid (AD, P_f\ ,\gamma) \overset{d}{=} DD \mid (AD,\ \gamma)$.

We have that

$$DD \mid (AD, P_f\ ,\gamma) \overset{d}{=} P(K_i = k_i, i = 1,2,\ldots,n+1 | K^* = k^*, K_0, P_f, \gamma)$$

$$= \frac{P(K_1 = k_1,\ldots,K_{n+1} = k_{n+1}, K^* = k^* | K_0; P_f, \gamma)}{P(K^* = k^* | K_0; P_f, \gamma)}$$

$$= \frac{P(K_1 = k_1,\ldots,K_{m-1} = k_{m-1}, K_m = k^* - k_1 - \ldots - k_{m-1}, K_{m+1} = k_{m+1},\ldots,K_{n+1} = k_{n+1} | K_0; P_f, \gamma)}{P(K^* = k^* | K_0; P_f, \gamma)}$$

$$= \frac{\binom{K_0}{k_1,\ldots,k_{n+1}} p_1^{k_1} p_2^{k_2} \cdots p_{m-1}^{k_{m-1}} p_m^{(k^* - k_1 - \ldots - k_{m-1})} p_{m+1}^{k_{m+1}} \cdots p_{n+1}^{k_{n+1}}}{\binom{K_0}{k^*} P_f^{\ k^*} (1 - P_f)^{K_0 - k^*}}$$

$$= \left[\binom{K_0}{k_1,\ldots,k_{n+1}} \Big/ \binom{K_0}{k^*} \right] \left(\frac{p_1}{P_f} \right)^{k_1} \cdots \left(\frac{p_{m-1}}{P_f} \right)^{k_{m-1}} \left(1 - \sum_{i=1}^{m-1} \frac{p_i}{P_f} \right)^{k^*-k_1-\ldots-k_{m-1}}$$

$$\cdot \left(\frac{p_{m+1}}{1-P_f} \right)^{k_{m+1}} \cdots \left(\frac{p_n}{1-P_f} \right)^{k_n} \left(1 - \sum_{i=m+1}^{n} \frac{p_i}{1-P_f} \right)^{k_{n+1}}$$

$$= \left[\binom{K_0}{k_1,\ldots,k_{n+1}} \Big/ \binom{K_0}{k^*} \right] \gamma_1^{k_1} \ldots \gamma_{m-1}^{k_{m-1}} \left(1 - \sum_{i=1}^{m-1} \gamma_i \right)^{k^*-k_1-\ldots-k_{m-1}}$$

$$\cdot \gamma_m^{k_{m+1}} \ldots \gamma_{n-1}^{k_n} \left(1 - \sum_{i=m}^{n-1} \gamma_i \right)^{k_{n+1}} \qquad (7.9)$$

$$= P(K_i = k_i,\, i = 1,2,\ldots,n+1 | K^* = k^*, K_0, \gamma)$$

$$\overset{d}{=} DD \,|\, (AD,\ \gamma).$$

Hence condition 7.6 holds. □

Note that $p_{n+1} = 1 - \sum_{i=1}^{n} p_i$, $k_{n+1} = K_0 - k_1 - \ldots - k_n$, and $k^* = k_1 + \ldots + k_m$.

Proposition 7.2 *The class of functions linking γ to DD |(AD, γ) includes all the powers of γ.*

Proof

By inspection of equation (7.9), one can see directly that the class of link functions $\gamma \to$ *DD* |(*AD*, γ) includes all the powers of γ. □

In theorems 7.2 and 7.4 below we give a necessary and sufficient conditions for perfect aggregation to hold for both Bernoulli system with dependent failures and the parallel Poisson systems that will be defined later. Previous work (Azaiez, 1993) was able to show only that the Dirichlet distribution is sufficient for perfect aggregation in the Bernoulli systems.

Theorem 7.2 *If g is the prior joint density of P, then perfect aggregation holds for our model if and only if g can be factored into a function of P_f and a function of γ (i.e., iff g possesses subcompositional invariance with respect to the partition defined by P_f and γ).*

Proof

Let g be the prior joint density of P. Then the joint density $h(P_f, \gamma)$ is given by

$$h(P_f, \gamma) = g(p_1, p_2, \ldots, p_n) \cdot \mathrm{J}(P \to (P_f, \gamma))$$

$$= g(P_f \gamma_1, P_f \gamma_2, \ldots, P_f \gamma_{m-1}, P_f (1 - \textstyle\sum_{i=1}^{m-1} \gamma_i), (1 - P_f)\gamma_m, \ldots,$$

$$(1 - P_f)\gamma_{n-1}) \cdot J(P_f), \qquad (*)$$

where $J(P_f) = \mathrm{J}(P \to (P_f, \gamma)) = P_f^{\,m-1}(1 - P_f)^{n-m}$.

By theorem 7.1 we have that perfect aggregation holds iff P_f and γ are independent. But P_f and γ are independent iff the joint density $h(P_f, \gamma)$ can be decomposed into a function of P_f and a function of γ. Since $J(P_f)$ is a function of P_f, then $h(P_f, \gamma)$ can be factored into a function of P_f and a function of γ iff g can be factored into a function of P_f and a function of γ. □

7.5 Distributions that satisfy perfect aggregation

In the last section we showed that perfect aggregation holds for our Bernoulli system if and only if the prior distribution of $P = (p_1, p_2, \ldots, p_n)$ can be factored into a function of P_f and a function of γ, where P_f and γ are as defined in (7.7) and (7.8), respectively (or equivalently, iff the prior distribution of $P = (p_1, p_2, \ldots, p_n)$ possesses subcompositional invariance with respect to the partition given by P_f and γ). In this section, using this result, we will give many examples of known prior distributions for $P=(p_1,p_2,...,p_n)$ that satisfy perfect aggregation. In section 7.5.1, we will give conditions on several known distributions such that perfect aggregation will hold for those distributions. Investigating the new family distributions defined in chapter 6 to identify particular distributions that satisfy perfect aggregation is left for our proposed research, as discussed in section 7.5.2.

7.5.1 Known distributions that satisfy perfect aggregation

As stated earlier, Azaiez (1993) showed that if the prior distribution of the system state probabilities is a Dirichlet distribution, then perfect aggregation holds. To illustrate our approach, we will show this again below.

Example 7.1 (*Dirichlet distribution*)

Let the prior joint density g of P be a Dirichlet density. Then

$$g(p_1,p_2,...,p_n)=\Gamma(\alpha^*)\prod_{i=1}^{n+1}\frac{p_i^{\alpha_i-1}}{\Gamma(\alpha_i)}$$

$$\sim\ (1-\textstyle\sum_{i=1}^{n}p_i)^{\alpha_{n+1}-1}\displaystyle\prod_{i=1}^{n}p_i^{\alpha_i-1}$$

$$=(1-P_f-(1-P_f)\textstyle\sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1}-1}P_f^{\sum_{i=1}^{m}\alpha_i-m}(1-P_f)^{\sum_{i=m+1}^{n}\alpha_i-(n-m)}\cdot$$

$$\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}\cdot(1-\textstyle\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\cdot\displaystyle\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}$$

$$=P_f^{\sum_{i=1}^{m}\alpha_i-m}(1-P_f)^{\sum_{i=m+1}^{n+1}\alpha_i-(n-m+1)}\cdot$$

$$\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}\cdot(1-\textstyle\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\cdot\displaystyle\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}\cdot(1-\textstyle\sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1}-1}$$

$=g_1(P_f)\cdot g_2(\gamma)$, where

$$g_1(P_f) = P_f^{\sum_{i=1}^{m}\alpha_i - m}(1-P_f)^{\sum_{i=m+1}^{n+1}\alpha_i - (n-m+1)}, \text{ and}$$

$$g_2(\gamma) = \prod_{i=1}^{m-1}\gamma_i^{\alpha_i - 1}\cdot(1-\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m - 1}\cdot\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}\cdot(1-\sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1}-1}.$$

Hence, perfect aggregation holds.

Remark 7.1

i) The joint density $h(P_f, \gamma)$ is given by

$$h(P_f,\gamma) \sim P_f^{m-1}(1-P_f)^{n-m} g_1(P_f)\cdot g_2(\gamma)$$

$$= \underbrace{[P_f^{\alpha_1^*-1}(1-P_f)^{\alpha_2^*-1}]}_{Be(\alpha_1^*,\alpha_2^*)}\cdot\underbrace{\left[\prod_{i=1}^{m-1}\gamma_i^{\alpha_i-1}(1-\sum_{i=1}^{m-1}\gamma_i)^{\alpha_m-1}\right]}_{D_m(\alpha_1,\alpha_2,...,\alpha_m)}\cdot\underbrace{\left[\prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1}-1}(1-\sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1}-1}\right]}_{D_{n-m+1}(\alpha_{m+1},\alpha_{m+2},...,\alpha_{n+1})}$$

where $\alpha_1^* = \sum_{i=1}^{m}\alpha_i$ and $\alpha_2^* = \sum_{i=m+1}^{n+1}\alpha_i$.

ii) Let $\boldsymbol{s}_1 = (\gamma_1, \gamma_2, ..., \gamma_{m-1})$ and $\boldsymbol{s}_2 = (\gamma_m, \gamma_{m+1}, ..., \gamma_{n-1})$. We will see that in all of our examples, $\{\boldsymbol{s}_1, \boldsymbol{s}_2, P_f\}$ are mutually independent (i.e., these priors possess partition independence), which is actually more than we need for perfect aggregation

Example 7.2 (*Connor-Mosimann distribution* $CM_n(\alpha,\lambda)$)

Let the prior joint density g of P be a generalized Dirichlet density as defined by Connor and Mosimann (1969). Then

$g(p_1, p_2, ..., p_n) = \prod_{i=1}^{n} B^{-1}(\alpha_i, \lambda_i) p_i^{\alpha_i - 1}(1-\sum_{j=1}^{i} p_j)^{\xi_i}$, where $\xi_i = \lambda_i - \alpha_{i+1} - \lambda_{i+1}$ for i =1,2,...,n-1 and $\xi_n = \lambda_n - 1$.

We want to show that perfect aggregation holds if and only if

$$\xi_i = 0 \text{ for } i=1,..,m-1. \quad (7.10)$$

Remember that states 1 through m in Ω are the states that cause the system to fail. Also, the order of the p_i is not arbitrary. We have

$$g(P) \sim \prod_{i=1}^{n} p_i^{\alpha_i - 1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i}$$

$$= (1 - \sum_{i=1}^{n} p_i)^{\xi_n} \prod_{i=1}^{n} p_i^{\alpha_i - 1} \cdot \prod_{i=1}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i}$$

$$= \underbrace{(1 - \sum_{i=1}^{n} p_i)^{\lambda_n - 1} \prod_{i=1}^{n} p_i^{\alpha_i - 1}}_{D_n(\alpha_1, \ldots, \alpha_n; \lambda_n)} \cdot \prod_{i=1}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i}$$

$$= g_1(P_f) \cdot g_2(\gamma) \cdot \prod_{i=1}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i} ,$$

where $\alpha_{n+1} = \lambda_n$, $g_1(P_f) = P_f^{\sum_{i=1}^{m}\alpha_i - m}(1 - P_f)^{\sum_{i=m+1}^{n+1}\alpha_i - (n-m+1)}$, and

$$g_2(\gamma) = \prod_{i=1}^{m-1}\gamma_i^{\alpha_i - 1} \cdot (1 - \sum_{i=1}^{m-1}\gamma_i)^{\alpha_m - 1} \cdot \prod_{i=m}^{n-1}\gamma_i^{\alpha_{i+1} - 1} \cdot (1 - \sum_{i=m}^{n-1}\gamma_i)^{\alpha_{n+1} - 1}.$$

It is not hard to see that $g(P)$ can be factored into a function of P_f and a function of γ if and only if $\prod_{i=1}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i}$ can be similarly factored. This can be done if and only if condition (7.10) holds.

To show sufficiency, assume that condition (7.10) holds. Then

$$\prod_{i=1}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i} = \prod_{i=m}^{n-1}(1 - \sum_{j=1}^{i} p_j)^{\xi_i}$$

$$= \prod_{i=m}^{n-1}(1 - P_f)^{\xi_i}(1 - \sum_{j=m}^{i-1}\gamma_j)^{\xi_i}$$

$$= \prod_{i=m}^{n-1}(1 - P_f)^{\xi_i} \cdot \prod_{i=m}^{n-1}(1 - \sum_{j=m}^{i-1}\gamma_j)^{\xi_i}$$