The covariances can then be specified as follows:

$$\mathrm{Cov}(x_i, x_j) = \begin{cases} Q_j \mathrm{E}(y_i) \prod_{m=1}^{i-1} \mathrm{E}(1 - y_m) & if \;\; i < k < j \\ Q_j \prod_{m=1}^{k-1} \mathrm{E}(1 - y_m) & if \;\; i = k < j \\ R_i \prod_{m=i+1}^{k-1} \mathrm{E}(1 - y_m) & if \;\; i < k = j \\ R_i \mathrm{E}(y_j) \prod_{m=i+1}^{j-1} \mathrm{E}(1 - y_m) & if \;\; i < j < k \\ \overline{R}_i \mathrm{E}(y_j) \prod_{m=i+1}^{j-1} \mathrm{E}(1 - y_m) & if \;\; k < i < j \end{cases} \tag{6.7}$$

Note that all terms in (6.7) are positive except for the Q_j, R_i, and $\overline{R_i}$, which may be either negative or positive independently. Thus, the upper triangular correlation matrix shown in Figure 6.1 below has three regions. In the rectangular region where $i \leq k < j$, either all correlations are negative or all correlations are positive (unlike in the case of the adaptive Dirichlet type B distribution, where all correlations in this region were required to be negative). There are also two triangular regions. Correlations in a given row of one of the triangular regions must have the same sign, but the signs may vary from row to row.

In chapter 4, we showed that the upper triangular correlation matrix obtained from the standard Liouville distribution on the positive orthant must be either completely positive or completely negative. Hence, one can see clearly that the type B extended Liouville distribution is substantially more flexible than the standard Liouville in modeling correlated data on the positive orthant.

Note that the type B extended Liouville distributions include the type C extended Liouville distributions (as the special case when k=1), which in turn include the standard Liouville distributions. Therefore, all the well-known distributions and their generalizations that are included in the standard Liouville family and discussed by Fang et al. (1990) are included in the family of type B extended Liouville distributions. These examples all have the Dirichlet distribution as a base, so they can be generalized easily by allowing a non-Dirichlet base. For example, analogous to the exponential-gamma Liouville distribution given by Fang et al., one could construct a class of new distributions on the n-dimensional unit simplex by letting f in equation (6.2) be an exponential-gamma density.

j

i

2	3	$\cdots$	$k-1$	k	$k+1$	$\cdots$	$n-2$	$n-1$	n	
s_1	s_1	$\cdots$	s_1	s_1	s_k	$\cdots$	s_k	s_k	s_k	1
	s_2	$\cdots$	s_2	s_2	s_k	$\cdots$	s_k	s_k	s_k	2
		$\cdots$	s_3	s_3	s_k	$\cdots$	s_k	s_k	s_k	3
			$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
				s_{k-1}	s_k	$\cdots$	s_k	s_k	s_k	$k-1$
					s_k	$\cdots$	s_k	s_k	s_k	k
						$\cdots$	s_{k+1}	s_{k+1}	s_{k+1}	$k+1$
							$\vdots$	$\vdots$	$\vdots$	$\vdots$
								s_{n-2}	s_{n-2}	$n-2$
									s_{n-1}	$n-1$

Figure 6.1. Correlation sign structure of the type B extended Liouville distribution, where $s_i \in \{-,+\}$ for $i = 1,\dots,n-1$.

The results for the special case k=1 (corresponding to the type C extended Liouville) can be deduced from the type B extended Liouville. However, as a convenience to the reader, we provide results for this case in the following section.

6.2.2 Type C extended Liouville distributions $EL_n^{(C)}(\alpha,\lambda; f)$

When k=1 in equation (6.2), this equation reduces to equation (6.8) below. This is because the double-cascaded topology reduces to a cascaded topology when k=1, in which case the type B adaptive Dirichlet reduces to the type C adaptive Dirichlet (i.e., Connor-Mosimann) distribution. For this, see Figures 3.2 and 3.3 in section 3.3.

Corollary 6.2 *The density function of a Type C extended Liouville distribution with generating density function* f *is given by*

$$g(x_1,\dots,x_n) = \prod_{i=1}^{n-1} \frac{\Gamma(\alpha_i+\lambda_i)}{\Gamma(\alpha_i)\Gamma(\lambda_i)} \frac{x_i^{\alpha_i-1}(x_{i+1}+\dots+x_n)^{\xi_i}}{(\sum_{i=1}^n x_i)^{\alpha_1+\lambda_1-1}} f\left(\sum_{i=1}^n x_i\right), \tag{6.8}$$

where $\xi_i = \lambda_i - \alpha_{i+1} - \lambda_{i+1}$ *for* $i = 1,\dots,n-2$, $\xi_{n-1} = \lambda_{n-1} - 1$.

Proof

The result follows by setting k=1 in equation (6.2). □

Remark 6.1

It is fairly easy to show that

i) If $\mathbf{x} \sim EL_n^{(C)}(\alpha,\lambda; f)$, $r<1$, and $f \sim Be(\lambda_n,\alpha_n)$, then $(1-\sum_{i=1}^n x_i, x_1, x_2,\dots,x_n)$ is distributed as $CM_{n+1}((\alpha_n,\alpha),(\lambda_n,\lambda))$ on H_{n+1}.

ii) If $x \sim EL_1^{(C)}(\alpha; f)$, then by equation (6.8) f is the density function of x.

Moments. The moments of the Connor-Mosimann distribution are given in section 2.1.4. Since r is independent of $\mathbf{z}$, we have

$$\begin{cases} \mathrm{E}(x_i) = \mathrm{E}(r)\mathrm{E}(z_i), & i = 1, \cdots, n; \\ \mathrm{Var}(x_i) = \mathrm{E}(r^2)\mathrm{E}(z_i^2) - [\mathrm{E}(r)]^2[\mathrm{E}(z_i)]^2, & i = 1, \cdots, n; \text{ and} \\ \mathrm{Cov}(x_i, x_j) = \mathrm{E}(r^2)\mathrm{E}(z_i z_j) - [\mathrm{E}(r)]^2 \mathrm{E}(z_i)\mathrm{E}(z_j), & i = 1, \cdots, n-1; \ j = i+1, \cdots, n. \end{cases} \tag{6.9}$$

Let $\alpha_n = 1$, $\lambda_n = 0$, and $M_{i-1} = \prod_{m=1}^{i-1}\left[(\lambda_m + 1)/(\alpha_m + \lambda_m + 1)\right]$ for $i = 1, \cdots, n$. Then we have the following:

$$\mathrm{E}(x_i) = \mathrm{E}(r)(\alpha_i/[\alpha_i + \lambda_i])\prod_{m=1}^{i-1}[\lambda_m/(\alpha_m + \lambda_m)], \quad i = 1, \cdots, n; \tag{6.10}$$

$$\mathrm{Var}(x_i) = \mathrm{E}(z_i)\{\mathrm{E}(r^2)[(\alpha_i + 1)/(\alpha_i + \lambda_i + 1)]M_{i-1} - [\mathrm{E}(r)]^2 \mathrm{E}(z_i)\}, \tag{6.11}$$
$$j = 1, \cdots, n;$$

$$\mathrm{Cov}(x_i, x_j) = \mathrm{E}(z_j)\{\mathrm{E}(r^2)[\alpha_i/(\alpha_i + \lambda_i + 1)]M_{i-1} - [\mathrm{E}(r)]^2 \mathrm{E}(z_i)\}, \tag{6.12}$$
$$i = 1, \cdots, n-1; \quad j = i+1, \cdots n.$$

$$\mathrm{E}(x_i^m) = \mathrm{E}(r^m)\{\prod_{j=1}^{i-1}[(\lambda_j)_m/(\alpha_j + \lambda_j)_m]\}[(\alpha_i)_m/(\alpha_i + \lambda_i)_m], \tag{6.13}$$
$$i = 2, \cdots, n-1.$$

where $(a)_m = a(a+1)\cdots(a+m-1)$.

Correlations. The sign of the R.H.S. of equation (6.12) does not depend on j, so the off-diagonal signs on any given row of the correlation matrix must be either all positive or all negative. Thus, this corresponds to Figure 6.1 in the limiting case when k=1 (so that the first k-1 rows and the first k columns of Figure 6.1 do not appear).

$$
\begin{array}{ccccc|c}
2 & 3 & \cdots & n-1 & n & \\
\hline
s_1 & s_1 & \cdots & s_1 & s_1 & 1 \\
 & s_2 & \cdots & s_2 & s_2 & 2 \\
 & & \cdots & s_3 & s_3 & 3 \\
 & & & \vdots & \vdots & \vdots \\
 & & & s_{n-2} & s_{n-2} & n-2 \\
 & & & & s_{n-1} & n-1
\end{array}
$$

Figure 6.2 Correlation sign structure of the type C extended Liouville distribution, where $s_i \in \{-,+\}$ for $i = 1, \ldots, n-1$.