Chapter 6

Extended Liouville distributions

As stated in chapter 1 and section 3.4, one of our concerns in this work is to identify a family of distributions that contains the Dirichlet but also contains distributions capable of modeling nontrivial dependence structures, as well as satisfying some of the other criteria given in section 1.1. This could be done both on the simplex and on the positive orthant. Results for the positive orthant will be given in this chapter, while results for the unit simplex will be given in chapter 7, particularly in section 7.5.2.

Section 6.1 delineates a conceptual framework for developing new positive orthant family of distributions; one actual extension is given in section 6.2. We will show below that both the Liouville and multiple Dickey distributions are contained in this new family. This gives us an intellectual framework within which to think about these special cases, which in turn provides a basis for selecting between these distributions. Also, our new family gives us more flexibility in modeling correlated data, since they can be shown to have significantly more general covariance and dependence structures than the standard Liouville distribution. Unfortunately, some of our new distributions are not analytically tractable. However, one can use Monte Carlo simulation to calculate moments and other performance measures.

Further motivation for introducing this new family of distributions is provided by the recent comment of Fang et al. (1990), who stated that "In analysing so-called ‘compositional data,’ scientists have been handicapped by the lack of known distributions to describe various patterns of variability." Aitchison (1986) has also discussed this problem. Fang et al. used the approach outlined in section 4.1 (with a Dirichlet distribution as a base) to come up with a new definition for the classical Liouville distribution. Part of our approach in developing more new positive orthant family of distributions is to allow for a non-Dirichlet base. In general, we will use the term ‘extended Liouville’ to refer to a Liouville distribution with a non-Dirichlet base.

6.1 The general idea on the positive orthant

Let $\mathbf{x} = (x_1,...,x_{n+1}) \in \Re_+^{n+1}$ and let $\mathbf{y} = \left(x_1/r,...,x_{n+1}/r\right)$, where $\sum_{i=1}^{n+1} x_i = r$ is a random variable. Then $\mathbf{x} = r \cdot \mathbf{y}$. We know from chapter 4 that if $\mathbf{y}$ follows the Dirichlet distribution, then the distribution of **x** is the standard Liouville distribution. Our approach here is to allow $\mathbf{y}$ to follow any one of the simplex distributions. One could call the resultant distribution a *Liouville distribution with* $\mathbf{y}$ *base*. For example, if $\mathbf{y}$ follows the multiple Dickey distribution, then the distribution of **x** could be called a *Liouville distribution with multiple Dickey base*, or a *multiple Dickey extended Liouville distribution*. Note that the Liouville and multiple Dickey distributions are (trivially) special cases of the multiple Dickey extended Liouville distribution. Also, since the adaptive Dirichlet family of distributions is a subclass of the multiple Dickey family of distributions, the multiple Dickey extended Liouville distribution is clearly quite general. This class of distributions will be discussed further in the next section.

The above illustrates our approach for developing new family of distributions on the positive orthant. As we said earlier, this gives us an intellectual framework within which to think about the special cases, which in turn provides a basis for selecting from among these distributions. Also, the new family gives us more flexibility in modeling correlated data. In the next section, we will document the covariance structures that can be obtained from some cases of the multiple Dickey extended Liouville distribution, as a basis for deciding whether this is a reasonable model for a particular data set.

6.2 Multiple Dickey extended Liouville distribution

Sivazlian (1981b) studied the standard Liouville distributions because of "their underlying generality, the interesting structured properties that they possess, and the better insight they provide into some well-known statistical theorems." Gupta and Richards (1987, 1990, 1991, 1992, 1995) have a series of articles studying the standard Liouville distribution. In particular, Gupta and Richards (1992) apply the Liouville distribution to reliability theory by studying a k-out-of-n system consisting of components whose lifetimes have a joint Liouville distribution. The multiple Dickey extended Liouville distribution will give us more flexibility in modeling such correlated data, since it has significantly more general covariance and dependence structures than the standard Liouville distribution.

Fang et al. (1990) devotes a complete chapter to the properties of the standard Liouville distribution. Some properties of the multiple Dickey extended Liouville distribution will be given in this section. Other properties can be established by paralleling the development given by Fang et al. for the standard Liouville distribution.

As mentioned in chapter 1, Gupta and Richards (1995) define a class of distributions, containing the classical Dirichlet and Liouville distributions, in which the random variables are defined on a locally compact Abelian group or semigroup. However, their work is mainly theoretical, and the authors do not give any application of their theory or test their distributions against the criteria we have suggested as desirable. Comparing our approach with theirs, we believe that their approach is less well suited for practical use than the one we propose here. Moreover, we have determined the correlation sign structures, for some distributions in our new family, while the sign structures obtainable using their approach are still unknown.

Definition 6.1 A random vector $\mathbf{x}$ in $\Re_+^n$ is said to have a ***multiple Dickey extended Liouville distribution*** if $\mathbf{x} \overset{d}{=} r\mathbf{z}$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n) \sim MD_n(\alpha, \beta, \boldsymbol{C})$ on $H_n = \{(z_1, \ldots, z_n) | \sum_{i=1}^n z_i = 1\}$ in $\Re_+^n$, and r is an independent r.v. with p.d.f. f ; in symbols, $\mathbf{x} \sim MDEL_n(\alpha, \beta, \boldsymbol{C}; f)$. We will call $\mathbf{z}$ the *multiple Dickey base,* $(\alpha, \beta, \boldsymbol{C})$ the *multiple Dickey parameters*, r the *generating variate*, and f the *generating density*.

It should be noticed immediately that when the multiple Dickey distribution reduces to the Dirichlet distribution, the multiple Dickey extended Liouville distribution reduces to the standard Liouville distribution on $\Re_+^n$, and when r=1 with probability one, the multiple Dickey extended Liouville distribution reduces to the multiple Dickey distribution $MD_n(\alpha, \beta, \boldsymbol{C})$ on H_n. Also, it is clear from the definition above that $\mathbf{x} \sim MDEL_n(\alpha, \beta, \boldsymbol{C}; f)$ if and only if the vector $(x_1 / \sum x_i, \ldots, x_n / \sum x_i) \sim MD_n(\alpha, \beta, \boldsymbol{C})$ on H_n and is independent of the total size $\sum x_i$ (equivalent to the random variable r in the definition of the multiple Dickey extended Liouville distribution).

The following theorem gives the density function of a multiple Dickey extended Liouville distribution:

Theorem 6.1 *The density function of a multiple Dickey extended Liouville distribution with generating density function* f *is given by*

$$g(x_1,\ldots,x_n) = A\,\frac{\prod_{i=1}^{n} x_i^{\alpha_i-1} \prod_{j=1}^{K} (\sum_{i=1}^{n} c_{ij} x_i)^{\beta_j}}{(\sum_{i=1}^{n} x_i)^{\alpha^*+\beta^*-1}} f(\sum_{i=1}^{n} x_i), \tag{6.1}$$

where $\alpha^* = \sum_{i=1}^{n} \alpha_i$, $\beta^* = \sum_{j=1}^{K} \beta_j$ *, and the other parameters are the same as given in the multiple Dickey distribution.*

Proof

The joint density of the independent random variables **z** and r is

$$A \prod_{i=1}^{n} z_i^{\alpha_i-1} \prod_{j=1}^{K} (\sum_{i=1}^{n} c_{ij} z_i)^{\beta_j} f(r).$$

Let $r = \sum_{i=1}^{n} x_i$, and let $z_j = x_j \big/ \sum_{i=1}^{n} x_i$ for $j = 1,\ldots,n-1$. The Jacobian of this transformation is r^{1-n} . Therefore, the p.d.f. of **x** is

$$A \prod_{i=1}^{n} (\frac{x_i}{\sum_{k=1}^{n} x_k})^{\alpha_i-1} \prod_{j=1}^{K} \left(\sum_{i=1}^{n} \left(c_{ij} \frac{x_i}{\sum_{k=1}^{n} x_k} \right) \right)^{\beta_j} f(\sum_{k=1}^{n} x_k)(\sum_{k=1}^{n} x_k)^{1-n}.$$

Simplifying this expression and letting $\alpha^* = \sum_{i=1}^{n} \alpha_i$ and $\beta^* = \sum_{j=1}^{K} \beta_j$ gives (6.1).□

Note that the density given by (6.1) is defined on the *a*-simplex $S_a^n = \{(x_1, x_2,\ldots,x_n) | 0 < \sum_{i=1}^{n} x_i < a\}$ if and only if f is defined on the interval (0,*a*). Also, since **z** and *r* are independent and $f(r)$ is itself a density function (and therefore integrates to one), the normalizing constant *A* for the multiple Dickey extended Liouville distribution is the same as the normalizing constant associated with the standard multiple Dickey distribution. Jiang et al. (1992) give computational methods for determining the constant *A* in some special cases where it has a closed-form expression. Some of these special cases will be discussed below.

In section 3.4, we showed that the adaptive Dirichlet family of distributions is a subclass of the multiple Dickey family. In section 6.2.1 below, we will concentrate on a special case of the multiple Dickey extended Liouville distribution where the multiple Dickey base is reduced to a type B adaptive Dirichlet distribution. The members of this subclass will be called type B extended Liouville distributions; in symbols, $EL_n^{(B)}(\alpha,\lambda,\mathrm{k};f)$. We will give the density function, moments, and correlation sign structure for distributions of this type. In section 6.2.2, we will consider the special case where *k*=1 (i.e., where the base is a Connor-Mosimann distribution). The members of this subclass will be called type C extended Liouville distribution; in symbols, $EL_n^{(C)}(\alpha,\lambda;f)$. These cases are considered because their moments and densities exhibit certain regularities that makes the moments computationally tractable and give us closed form for the densities.

6.2.1 Type B Extended Liouville distributions $EL_n^{(B)}(\alpha,\lambda,\mathrm{k};f)$

Let $\mathbf{z} \sim AD_n^{(B)}(\alpha,\lambda,\mathrm{k})$ on H_n be the base in definition 6.1. The density function of the resultant extended Liouville distribution is given below:

Corollary 6.1 *The density function of a Type B extended Liouville distribution with generating density function f is given by*

$$g_k(\mathbf{x}) = A\left[x_k^{\lambda_{k-1}-1}\prod_{i=1}^{k-1}x_i^{\alpha_i-1}(\sum_{j=i}^{k}x_j)^{\lambda_{i-1}-\alpha_i-\lambda_i}\right]\left[x_n^{\lambda_{n-1}-1}\prod_{i=k+1}^{n-1}x_i^{\alpha_i-1}(\sum_{j=i}^{n}x_j)^{\lambda_{i-1}-\alpha_i-\lambda_i}\right]\frac{f(\sum_{i=1}^{n}x_i)}{(\sum_{i=1}^{n}x_i)^{\alpha_k+\lambda_k-1}} \tag{6.2}$$

where $A = \prod_{i=1}^{n-1}\frac{\Gamma(\alpha_i+\lambda_i)}{\Gamma(\alpha_i)\Gamma(\lambda_i)}$ *and* $\lambda_0 = \alpha_k$.

Proof

If f is the density of r and $\mathbf{z} \sim AD_n^{(B)}(\alpha, \lambda, \mathrm{k})$, then following the same steps as in the proof of theorem 6.1, we get the p.d.f of $\mathbf{x}$ as given in (6.2). Also, one can prove this corollary by remembering that the type B adaptive Dirichlet is a special case of the multiple Dickey distribution and then using theorem 6.1. □

Moments. The moments of $\mathbf{z} \sim AD_n^{(B)}(\alpha, \lambda, \mathrm{k})$ are given in section 3.3. Since r is independent of $\mathbf{z}$, we have

$$\begin{cases} \mathrm{E}(x_i) = \mathrm{E}(r)\mathrm{E}(z_i), & i = 1, \cdots, n; \\ \mathrm{Var}(x_i) = \mathrm{E}(r^2)\mathrm{E}(z_i^2) - [\mathrm{E}(r)]^2[\mathrm{E}(z_i)]^2, & i = 1, \cdots, n; \text{ and} \\ \mathrm{Cov}(x_i, x_j) = \mathrm{E}(r^2)\mathrm{E}(z_i z_j) - [\mathrm{E}(r)]^2 \mathrm{E}(z_i)\mathrm{E}(z_j) & i = 1, \cdots, n-1; \ j = i+1, \cdots, n. \end{cases} \tag{6.3}$$

Covariances. To clarify the correlation sign structures, let y_i be the ratios associated with the vector of fractions $\mathbf{z}$, and define

$$Q_j = \mathrm{E}(y_j) \prod_{m=k+1}^{j-1} \mathrm{E}(1 - y_m)[\mathrm{E}(r^2)\mathrm{E}(y_k(1-y_k)) - \mathrm{E}^2(r)\mathrm{E}(y_k)\mathrm{E}(1-y_k)],$$
$$j = k+1, \dots, n, \tag{6.4}$$

$$R_i = \mathrm{E}(r^2)\mathrm{E}(y_k^2)\mathrm{E}(y_i(1-y_i))\prod_{m=1}^{i-1}\mathrm{E}((1-y_m)^2)$$
$$- \mathrm{E}^2(r)\mathrm{E}^2(y_k)\mathrm{E}(y_i)\mathrm{E}(1-y_i)\prod_{m=1}^{i-1}\mathrm{E}^2(1-y_m), \ i = 1, \dots, k-1, \tag{6.5}$$

and

$$\overline{R}_i = \mathrm{E}(r^2)\mathrm{E}(y_i(1-y_i))\prod_{m=k}^{i-1}\mathrm{E}((1-y_m)^2) - \mathrm{E}^2(r)\mathrm{E}(y_i)\mathrm{E}(1-y_i)\prod_{m=k}^{i-1}\mathrm{E}^2(1-y_m),$$
$$i = k+1, \dots, n. \tag{6.6}$$